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Homogeneous functions

A function z is said to be homogeneous function of x and y if it can be written in form of $x^n \cdot f\left(\frac{y}{x}\right)$

$$\text{ie } z = f(x,y) = x^n \cdot f\left(\frac{y}{x}\right)$$

or

$$y^n f\left(\frac{x}{y}\right).$$

Here n is the degree of homogeneous function.

Eg i) $f(x,y) = \dfrac{x^3+y^3}{x+y}$.

$$= \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1+\frac{y}{x}\right]} = x^2 f\left(\frac{y}{x}\right)$$

Hence it is a homogeneous function of degree 2

ii) $f(x,y) = \dfrac{\sqrt{x}+\sqrt{y}}{x+y}$

$$= \frac{\sqrt{x}\left[1+\sqrt{\frac{y}{x}}\right]}{x\left[1+\frac{y}{x}\right]} = x^{-1/2} f\left(\frac{y}{x}\right)$$

homogeneous function of degree $-1/2$.

iii) $f(x,y,z) = 3x^2yz + 5xy^2z + 4z^4$.

$$x^4\left[3\frac{y}{x}\cdot\frac{z}{x} + 5\left(\frac{y}{x}\right)^2\frac{z}{x} + 4\cdot\left(\frac{z}{x}\right)^4\right] = x^4 \cdot f\left(\frac{y}{x}, \frac{z}{x}\right)$$

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Hence it is a homogeneous function of degree 4.

Euler's Theorem of Homogeneous functions

If z is a homogeneous function of x and y of degree n then $x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = nz$.

Proof:- Since z is a homogeneous function of degree n

$$\therefore \quad z = x^n \cdot f\left(\frac{y}{x}\right).$$

$$\frac{\partial z}{\partial x} = nx^{n-1}\cdot f\left(\frac{y}{x}\right) + x^n \cdot f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$ (By Product Rule of differentiation)

$$\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$ ($\because$ Here x^n is a constant)

LHS

$$x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = x\left[nx^{n-1}\cdot f\left(\frac{y}{x}\right) + x^n\cdot f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)\right] + y\left[x^n\cdot f'\left(\frac{y}{x}\right)\cdot\frac{1}{x}\right]$$

$$= nx^n f\left(\frac{y}{x}\right) + x^{n+1-2} f'\left(\frac{y}{x}\right)(-y) + yx^{n-1} f'\left(\frac{y}{x}\right).$$

$$= nz - x^{n-1} f'\left(\frac{y}{x}\right) y + yx^{n-1}\cdot f'\left(\frac{y}{x}\right)$$

$$= nz$$

Hence proved.

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ii) If z is homogeneous function of x and y of degree n then ~~prove that~~

$$\boxed{x^2 \cdot \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z}$$

Practice Questions

i) All examples are important.

ii) If $z = \log(x^2+xy+y^2)$ then show that

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2.$$

Proof $z = \log(x^2+xy+y^2)$.

$$e^z = e^{\log(x^2+xy+y^2)}$$

$$\Rightarrow e^z = x^2+xy+y^2 \qquad (\because e^{\log x} = x.)$$

Let $u = e^z = x^2+xy+y^2$

$$= x^2\left[1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2\right] = x^2 f\left(\frac{y}{x}\right)$$

Hence $u = e^z$ is a homogeneous function of degree 2.

By Euler's theorem

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 2u. \qquad -(1)$$

$$\frac{\partial u}{\partial x} = e^z \cdot \frac{\partial z}{\partial x} \qquad \frac{\partial u}{\partial y} = e^z \frac{\partial z}{\partial y}.$$

Put these values and $u = e^z$ in eq. (1).

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$$x e^z \frac{\partial z}{\partial x} + y e^z \frac{\partial z}{\partial y} = 2e^z$$

$$e^z \left(x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} \right) = 2e^z$$

$$\Rightarrow x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2.$$

iii) If $z = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \frac{1}{4} \sin 2z.$$

Proof :- Given $z = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\tan z = \frac{x+y}{\sqrt{x}+\sqrt{y}}.$$

Let $u = \tan z = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left[1+\frac{y}{x}\right]}{\sqrt{x}\left[1+\sqrt{\frac{y}{x}}\right]} = \sqrt{x} \cdot f\left(\frac{y}{x}\right)$

Hence u is a homogeneous function of degree $1/2$

By Euler's theorem

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu. \quad \text{Here } n = 1/2 \qquad \text{--- (1)}$$

Now $u = \tan z$

$$\frac{\partial u}{\partial x} = \sec^2 z \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial y} = \sec^2 z \cdot \frac{\partial z}{\partial y}$$

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Put these values in eq. (1)

$$x \cdot \sec^2 z \cdot \frac{\partial z}{\partial x} + y \cdot \sec^2 z \frac{\partial z}{\partial y} = \frac{1}{2} \cdot \tan z$$

$$\sec^2 z \left[x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} \right] = \frac{1}{2} \tan z$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \frac{1}{2} \cdot \frac{\tan z}{\sec^2 z}$$

$$= \frac{1}{2} \frac{\sin z}{\cos z} \cos^2 z \cdot$$

$$= \frac{1}{2} \sin z \cdot \cos z \cdot$$

$$= \frac{1}{2} \cdot \frac{2}{2} \sin z \cos z = \frac{1}{4} \sin 2z$$

$(\because \sin 2A = 2 \cos A \sin A)$

iii) If $u = \frac{x^2 y^2}{x+y}$ Prove that

$$x \cdot \frac{\partial^2 u}{\partial x^2} + y \cdot \frac{\partial^2 u}{\partial x \partial y} = 2 \frac{\partial u}{\partial x}.$$

Proof $u = \frac{x^2 y^2}{x+y} = \frac{x^4 \left[\frac{y^2}{x^2} \right]}{x \left[1 + \frac{y}{x} \right]} = x^3 \cdot f\left(\frac{y}{x}\right)$

Hence u is a homogeneous function of degree 3.

By Euler's theorem

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 3 \cdot u \cdot \quad \text{—(1)}$$

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Partially differentiate eq ① wrt x.

$$x \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \cdot \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} \cdot \underbrace{0}_{=0} = 3\frac{\partial u}{\partial x}$$

$$x \cdot \frac{\partial^2 u}{\partial x^2} + y \cdot \frac{\partial^2 u}{\partial x \partial y} = 3\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}$$

$$\boxed{x\frac{\partial^2 u}{\partial x^2} + y \cdot \frac{\partial^2 u}{\partial x \partial y} = 2\frac{\partial u}{\partial x}.}$$

Hence proved.

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